Advanced Mechanics

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Exam, January 31, 2013.
14:00-17:00, 5161,0293
4 problems (total of 51 points).
The solution of every problem on a separate piece of paper with name and student number.
Some useful formulas are listed at the end.

Problem 1 (13 pnts in total)

A mass M is sliding down a frictionless slope that makes an angle α with the horizontal plane. A pendulum consisting of a string of length d and mass m hangs from M

1 pnts a. Make a figure which clearly shows the coordinates you use for solving this problem.

4 pnts b. Write down the Lagrangian for the system.

2 pnts c. Determine the Euler-Lagrange equations of motion.

2 pnts d. Show that for $M \gg m$ this can be reduced to (NOTE: the signs will depend on your definition of coordinates and may thus differ)
$g \sin\alpha = \ddot{l}$, and $g \sin\theta = g \sin\alpha \cos(\alpha - \theta) - d\ddot{\theta}$,
where θ is the angle of the string with the vertical and l is the distance along the slope.

2 pnts e. Find the angle of the string for which the mass m is at rest with respect to the mass M (the mass m does only move linearly and does not follow a swinging motion).

2 pnts f. Find the frequency of small oscillations of m for the case that $M \gg m$.

Problem 2 (15 pnts in total)

Consider the eom of a damped oscillator which is driven by a time-dependent force, $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F(t)/m$.

2 pnts a. Give the complete expression for the Greens function $G(t, t')$ for the oscillator of this problem. Hint: look at the formulas at the end of this exam.

3 pnts b. Show, by direct substitution, that the Greens function solves the eom for $F(t) = 0$ at all times except $t = t'$.

4 pnts c. Give the explicit expression for $x(t)$ in terms of a definite integral for the case that

$$F(t) = \begin{cases} 0 & t < -\tau/2 \\ (a/\tau)t & -\tau/2 < t < \tau/2 \\ 0 & t > \tau/2 \end{cases} .$$

Distinguish three cases: $t < -\tau/2$, $-\tau/2 < t < \tau/2$, and $t > \tau/2$.

4 pnts d. Solve for $x(t)$ for $t > \tau/2$ in the limit $\beta = 0$.

2 pnts e. What do you expect for $x(t)$ in the limit $\omega_0\tau \ll 1$ and give a short explanation.

Problem 3 (10 pnts in total)

A heavy mass m is hanging on a long string of length l suspended from a high ceiling on Earth at a latitude λ (note, the Equator corresponds to $\lambda = 0°$, the North pole to $\lambda = 90°$). Use a local coordinate system where $\hat{x}$=South; $\hat{y}$=East; $\hat{z}$ along $-\vec{g}$= vertical up along a plumb line. The Earth is rotating with an angular frequency ω_E. The mass is swinging with a small amplitude. Ignore friction.

2 pnts a. Express the rotation vector of the Earth in the local coordinate system. Show that this expression has the right limits for the North pole, the Equator and the South pole.

3 pnts b. Show that the expression for the acceleration of the mass can be expressed as

$$\ddot{x} = -\omega_0^2 x + 2\omega_E \sin\lambda\, \dot{y}$$

$$\ddot{y} = -\omega_0^2 y - 2\omega_E \sin\lambda\, \dot{x}$$

and express ω_0 in terms of m, l, and g.

3 pnts c. Give the solution for the case of small amplitude motion. How fast does the plane of the oscillator rotate? give also the values at the North pole and at the Equator.

2 pnts d. What ratio should be small to call this a 'small amplitude oscillation'. At what point is this important for this problem and why?

Problem 4 (13 pnts in total)

Find the shortest path on a surface given by $z = x^{3/2}$ between the two points $P_1 = (0,0,0)$ and $P_2 = (1,1,1)$.

3 pnts a. Express the path length as an integral along the curve.

2 pnts b. Give the resulting Euler equation(s) for the path.

3 pnts c. Show that the solution of the Euler equation can be written as $y(x) = A(1 + \frac{9x}{4})^{\frac{3}{2}} + B$.

2 pnts d. Determine the constants A and B for the path between P_1 and P_2 (do not expect round numbers).

3 pnts e. Determine the path length between P_1 and P_2.

Possibly useful formulas:

$\vec{F}' = \vec{F}_{\text{inert}} - 2m\vec{\omega} \times \vec{v}' - m\dot{\vec{\omega}} \times \vec{r}' - m\vec{\omega} \times (\vec{\omega} \times \vec{r}')$, and $\vec{v} = \vec{v}' + \vec{\omega} \times \vec{r}'$

$$\int x e^{cx} dx = \frac{e^{cx}}{c^2}(cx - 1)$$

The response of a damped oscillator $\ddot{x} + 2\beta\dot{x} + \omega_r^2 x = F(t)/m$ to a delta force at $t = 0$ is $\frac{1}{\omega_1 m} e^{-\beta t} \sin\omega_1 t$ for $t > 0$, where $\omega_1 = \sqrt{\omega_r^2 - \beta^2}$.

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$; $\cos(\alpha - \beta) = \sin\alpha\sin\beta + \cos\alpha\cos\beta$